

Dominated Splitting and Pesin's Entropy Formula

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Abstract

Let M be a compact manifold and $f : M \rightarrow M$ be a C^1 diffeomorphism on M . If μ is an f -invariant probability measure which is absolutely continuous relative to Lebesgue measure and for μ a. e. $x \in M$, there is a dominated splitting $T_{orb(x)}M = E \oplus F$ on its orbit $orb(x)$, then we give an estimation through Lyapunov characteristic exponents from below in Pesin's entropy formula, i.e., the metric entropy $h_\mu(f)$ satisfies

$$h_\mu(f) \geq \int \chi(x) d\mu,$$

where $\chi(x) = \sum_{i=1}^{dim F(x)} \lambda_i(x)$ and $\lambda_1(x) \geq \lambda_2(x) \geq \dots \geq \lambda_{dim M}(x)$ are the Lyapunov exponents at x with respect to μ . Consequently, by using a dichotomy for generic volume-preserving diffeomorphism we show that Pesin's entropy formula holds for generic volume-preserving diffeomorphisms, which generalizes a result of Tahzibi [12] in dimension 2.

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1 Introduction

To estimate metric entropy through Lyapunov exponents is an important topic in differential ergodic theory. In 1977 Ruelle [11] got from above an estimate of metric entropy of an invariant measure, and Pesin[10] in 1978 got from below an estimation of metric entropy of an invariant measure absolutely continuous relative to Lebesgue measure and thus got a so called Pesin' entropy formula. Pesin's proof is based on the stable manifold theorem. In 1980 Mañé [7] gave another ingenious and very simple proof without using the theory of stable manifolds. In 1985 Ledrappier and Young[4] generalized the formula to all SRB measures, not necessarily absolutely continuous relative to Lebesgue measure. There are also more generalizations[5, 6].

Pesin's entropy formula by Pesin and by Mañé and by others assumes that not only the differentiability of the given dynamics is of class C^1 but also that the first derivative satisfies an α -Hölder condition for some $\alpha > 0$. It is interesting to investigate Pesin's entropy formula under the weaker C^1 differentiability hypothesis plus some additional condition, for example, dominated splitting. The aim of this paper is to prove that Pesin's entropy formula remains true for invariant probability measure absolutely continuous relative to Lebesgue measure in the C^1 diffeomorphisms with dominated splitting. In the proof of [7], the combination of the graph transform method (Lemma 3 there) and the distortion property deduced from the Hölder condition of the derivative play important roles. The domination assumption in our C^1 diffeomorphism helps us to overcome much trouble. Our proof follows Mañé without using the theory of stable manifolds, as noted by Katok that it seems that Mañé's proof can also be extended to the more general framework.

Tahzibi showed in [12] that there is a residual subset \mathcal{R} in C^1 volume-preserving **surface** diffeomorphisms such that every system in \mathcal{R} satisfies Pesin's entropy formula. As an consequence our main Theorem 2.2 and a result of Bochi and Viana[2], we generalize the result of Tahzibi into any dimensional case.

2 Results

Before stating our main results we need to introduce the concept of dominated splitting. Denote the minimal norm of a linear map A by $m(A) = \|A^{-1}\|^{-1}$.

Definition 2.1. Let $f : M \rightarrow M$ be a C^1 diffeomorphism on a compact Riemannian manifold.

(1). (Dominated splitting at one point) Let $x \in M$ and $T_{\text{orb}(x)}M = E \oplus F$ be a Df -invariant splitting on $\text{orb}(x)$. $T_{\text{orb}(x)}M = E \oplus F$ is called to be $N(x)$ -dominated at x , if there exists a constant $N(x) \in \mathbb{Z}^+$ such that

$$\frac{\|Df^{N(x)}|_{E(f^j(x))}\|}{m(Df^{N(x)}|_{F(f^j(x))})} \leq \frac{1}{2}, \quad \forall j \in \mathbb{Z}.$$

(2). (Dominated splitting on an invariant set) Let Δ be an f -invariant set and $T_\Delta M = E \oplus F$ be a Df -invariant splitting on Δ . We call $T_\Delta M = E \oplus F$ to be a N -dominated splitting, if there exists a constant $N \in \mathbb{Z}^+$ such that

$$\frac{\|Df^N|_{E(y)}\|}{m(Df^N|_{F(y)})} \leq \frac{1}{2}, \quad \forall y \in \Delta.$$

For a Borel measurable map $f : M \rightarrow M$ on a compact metric space M and an f -invariant measure μ , we denote by $h_\mu(f)$ the metric entropy.

Now we state our results as follows.

Theorem 2.2. Let $f : M \rightarrow M$ be a C^1 diffeomorphism on a compact Riemannian manifold. Let f preserve an invariant probability measure μ which is absolutely continuous relative to Lebesgue measure. For μ a.e. $x \in M$, denote by

$$\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_{\dim M}(x)$$

the Lyapunov exponents at x . Let $m(\cdot) : M \rightarrow \mathbb{N}$ be an f -invariant measurable function. If for μ a.e. $x \in M$, there is a $m(x)$ -dominated splitting: $T_{\text{orb}(x)}M = E_{\text{orb}(x)} \oplus F_{\text{orb}(x)}$, then

$$h_\mu(f) \geq \int \chi(x) d\mu,$$

where $\chi(x) = \sum_{i=1}^{\dim F(x)} \lambda_i(x)$.

In particular, if for μ a.e. $x \in M$, $E(x)$ and $F(x)$ coincide with the sum of the Oseledec subbundles corresponding to negative Lyapunov exponents and non-negative Lyapunov exponents respectively (or, $E(x)$ corresponds to non-positive Lyapunov exponents and $F(x)$ corresponds to positive Lyapunov exponents), then

$$h_\mu(f) = \int \chi(x) d\mu = \int \sum_{\lambda_i(x) \geq 0} \lambda_i(x) d\mu.$$

In other words, Pesin's entropy formula holds.

Remark. Recall that the well known Ruelle's inequality[11]

$$h_\mu(f) \leq \int \sum_{\lambda_i(x) \geq 0} \lambda_i(x) d\mu$$

is valid for any invariant measure of f . Thus, if the inverse inequality hold, the particular case of Theorem 2.2 is deduced immediately. So the left work we need to prove is the inverse inequality.

Since Yang have proved in [13] that for any diffeomorphism f far away from homoclinic tangency and any f -ergodic measure μ , the sum of the stable, center and unstable bundles in Oseledec splitting is dominated on $\text{supp}(\mu)$, using Theorem 2.2 we have a direct corollary as follows.

Corollary 2.3. *Let $f \in \text{Diff}^1(M)$ far away from homoclinic tangency and let μ be an f -ergodic probability measure which is absolutely continuous relative to Lebesgue measure. Then f satisfies Pesin's entropy formula, i.e.,*

$$h_\mu(f) = \sum_{\lambda_i > 0} \lambda_i,$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\dim M}$ are the Lyapunov exponents with respect to μ .

Let m be the volume measure and $\text{Diff}_m^1(M)$ denote the space of volume-preserving diffeomorphisms. It is known that the stable bundle and unstable bundle of Anosov diffeomorphism are always dominated, and so are the bundles between the stable, center and unstable directions in partially hyperbolic systems. Thus we have a direct corollary as follows.

Corollary 2.4. *Let $f \in \text{Diff}_m^1(M)$. If f is an Anosov diffeomorphism (or, a partially hyperbolic diffeomorphism which satisfies that for m a.e. x , the Lyapunov exponents at x in the central bundle are either all non-positive or all non-negative), then Pesin's entropy formula holds.*

In a Baire space, we say a set is residual if it contains a countable intersection of dense open sets. We always call every element in the residual set to be a generic point. It is known that every $C^{1+\alpha}$ volume-preserving diffeomorphism satisfies Pesin's entropy formula(see [7, 10]) and the set of $C^{1+\alpha}$ (or C^2) volume-preserving diffeomorphisms is dense in $\text{Diff}_m^1(M)$, so the set of

volume-preserving diffeomorphisms satisfying Pesin's entropy formula is dense in $\text{Diff}_m^1(M)$. Hence, it is natural to ask whether generic volume-preserving diffeomorphisms satisfy Pesin's entropy formula. This problem is not trivial because A. Tahzibi showed in [12] that $C^{1+\alpha}$ volume-preserving diffeomorphisms are not generic in $\text{Diff}_m^1(M)$. Here we use Theorem 2.2 to deduce this generic property.

Theorem 2.5. *There exists a residual subset $\mathcal{R} \subseteq \text{Diff}_m^1(M)$ such that for every $f \in \mathcal{R}$, the metric entropy $h_\mu(f)$ satisfies Pesin's entropy formula, i.e.,*

$$h_\mu(f) = \int \sum_{\lambda_i(x) \geq 0} \lambda_i(x) dm,$$

where $\lambda_1(x) \geq \lambda_2(x) \geq \dots \geq \lambda_{\dim M}(x)$ are the Lyapunov exponents of x with respect to m .

Remark. If $\dim(M) = 2$, this result is firstly proved in [12].

3 Proof of Theorem 2.2

Our proof will be based on a general lower estimate for metric entropy, which makes it possible to avoid the use of partitions. Let $g : M \rightarrow M$ be a map, d be a metric on M and let $\delta > 0$. If $x \in M$ and $n \geq 0$, define Bowen ball

$$B_n(g, \delta, x) = \{y \in M \mid d(g^j(x), g^j(y)) \leq \delta, 0 \leq j \leq n\}.$$

In other words,

$$B_n(g, \delta, x) = \bigcap_{j=0}^n g^{-j} B_\delta(g^j(x)),$$

where $B_\delta(g^j(x))$ denotes the ball centered at x with radius δ . If $g : M \rightarrow M$ is measurable and μ is a measure on M (not necessarily g -invariant), define

$$h_\mu(g, \delta, x) = \limsup_{n \rightarrow +\infty} \frac{1}{n} [-\log \mu(B_n(g, \delta, x))].$$

Lemma 3.1. *If g is measurable, μ is a g -invariant probability measure on M and $\nu \gg \mu$ is another measure on M (not necessarily g -invariant), then*

$$h_\mu(g) \geq \sup_{\delta > 0} \int_M h_\nu(g, \delta, x) d\mu.$$

Proof This lemma is a particular case of the Proposition in [7], see P.96 or Lemma 13.4 in [8] for details. \square

Before going into the proof of Pesin's formula we shall prove a technical lemma. The reader familiar with the Hadamard graph transform method for constructing invariant manifolds will recognize this lemma one of the steps of that method. In the statement of the lemma we shall use the following definitions from [7, 8].

Definition 3.2. Let E be a normed space and $E = E_1 \oplus E_2$ be a splitting. Define $\gamma(E_1, E_2)$ as the supremum of the norms of the projections $\pi_i : E \rightarrow E_i$ $i = 1, 2$, associated with the splitting. Moreover, we say that a subset $G \subset E$ is a (E_1, E_2) -graph if there exists an open $U \subseteq E_2$ and a C^1 map $\psi : U \rightarrow E_1$ satisfying

$$G = \{x + \psi(x) \mid x \in U\}.$$

The number $\sup\{\frac{\|\psi(x) - \psi(y)\|}{\|x - y\|} \mid x \neq y \in U\}$ is called the dispersion of G .

The following lemma about graph transform on dominated bundles is a generalization to Lemma 3 in Mañé[7] about that on hyperbolic bundles. Observe that the main point of the proof of Lemma 3 there is the gap between two hyperbolic bundles and can be replaced by the gap of two dominated bundles, our proof of the following lemma is a slight change of the proof of Lemma 3 in Mañé[7]. We give a proof for completeness.

Lemma 3.3. Given $\alpha > 0$, $\beta > 0$ and $c > 0$, there exists $\tau > 0$ with the following property. If E is a finite-dimensional normed space and $E = E_1 \oplus E_2$ a splitting with $\gamma(E_1, E_2) \leq \alpha$, and \mathcal{F} is a C^1 embedding of a ball $B_\delta(0) \subset E$ into another Banach space E' satisfying

- (i). $D_0\mathcal{F}$ is an isomorphism and $\gamma((D_0\mathcal{F})E_1, (D_0\mathcal{F})E_2) \leq \alpha$;
- (ii). $\|D_0\mathcal{F} - D_x\mathcal{F}\| \leq \tau$ for all $x \in B_\delta(0)$;
- (iii). $\frac{\|D_0\mathcal{F}|_{E_1}\|}{m(D_0\mathcal{F}|_{E_2})} \leq \frac{1}{2}$;
- (iv). $m(D_0\mathcal{F}|_{E_2}) \geq \beta$;

then for every (E_1, E_2) -graph G with dispersion $\leq c$ contained in the ball $B_\delta(0)$, its image $\mathcal{F}(G)$ is a $((D_0\mathcal{F})E_1, (D_0\mathcal{F})E_2)$ -graph with dispersion $\leq c$.

Proof Identity E with $E_1 \times E_2$ and E' with $(D_0\mathcal{F})E_1 \times (D_0\mathcal{F})E_2$. Write the map \mathcal{F} in the form

$$\mathcal{F}(x, y) = (Lx + p(x, y), Ty + q(x, y)),$$

where $L = (D_0\mathcal{F})E_1$, $T = (D_0\mathcal{F})E_2$. It follows that the partial derivatives of p and q with respect to x and y have norm $\leq \tau\alpha$.

Let $U \subset E_2$ be an open set and $\psi : U \rightarrow E_1$ a map whose graph $\{(\psi(v), v) | v \in U\}$ is G . Then,

$$\mathcal{F}(G) = \{(L\psi(v) + p(\psi(v), v), Tv + q(\psi(v), v)) | v \in U\}.$$

To study this set define $\phi : U \rightarrow (D_0\mathcal{F})E_2$ by

$$\phi(v) = Tv + q(\psi(v), v).$$

If $v, w \in U$,

$$\|\phi(v) - \phi(w)\| \geq \|T(v - w)\| - \|q(\psi(v), v) - q(\psi(w), w)\|.$$

Using the fact that the norm of the partial derivatives of q are $\leq \tau\alpha$ and hypothesis (iii) we obtain

$$\begin{aligned} \|\phi(v) - \phi(w)\| &\geq m(T)\|v - w\| - \tau\alpha(\|\psi(v) - \psi(w)\| + \|v - w\|) \\ &\geq (m(T) - \tau\alpha(1 + c))\|v - w\|. \end{aligned}$$

Hence, if τ is so small that

$$m(T) - \tau\alpha(1 + c) \geq \beta - \tau\alpha(1 + c) > 0,$$

ϕ is a homeomorphism of U onto $\phi(U)$ whose inverse has Lipschitz constant $\leq (\beta - \tau\alpha(1 + c))^{-1}$. In particular, $\phi(U)$ is open. Now define $\hat{\psi} : \phi(U) \rightarrow (D_0\mathcal{F})E_1$ by

$$\hat{\psi}(v) = (L\psi\phi^{-1})(v) + p(\psi(\phi^{-1}(v)), \phi^{-1}(v)).$$

Clearly,

$$\mathcal{F}(G) = \{(\hat{\psi}(x), x) | x \in \phi(U)\}.$$

To calculate the dispersion of $\mathcal{F}(G)$, write

$$\hat{\psi} = \tilde{\psi}\phi^{-1}$$

where $\tilde{\psi}(w) = L\psi(w) + p(\psi(w), w)$. Then

$$\begin{aligned} \|\tilde{\psi}(w) - \tilde{\psi}(v)\| &\leq \|L\|\|\psi(v) - \psi(w)\| + \tau\alpha(\|\psi(v) - \psi(w)\| + \|v - w\|) \\ &\leq (c\|L\| + \tau\alpha(1 + c))\|v - w\|. \end{aligned}$$

Then the dispersion of $\mathcal{F}(G)$ is less than or equal to

$$\begin{aligned} c \frac{\|L\| + \tau\alpha(1+c)/c}{m(T) - \tau\alpha(1+c)} &\leq c \frac{\frac{1}{2}m(T) + \tau\alpha(1+c)/c}{m(T) - \tau\alpha(1+c)} \\ &= c \frac{\frac{1}{2} + \tau\alpha(1+c)/cm(T)}{1 - \tau\alpha(1+c)/m(T)} \leq c \frac{\frac{1}{2} + \tau\alpha(1+c)/c\beta}{1 - \tau\alpha(1+c)/\beta}. \end{aligned}$$

Taking τ small enough, the factor of c is < 1 and the lemma is proved. \square

Lemma 3.4. *Let $g \in \text{Diff}^1(M)$ and Λ be g -invariant subset of M . If there is a 1-dominated splitting on Λ : $T_\Lambda M = E \oplus F$, then for any $c > 0$, there exists $\delta > 0$ such that for every $x \in \Lambda$ and any (E_x, F_x) -graph G with dispersion $\leq c$ contained in Bowen ball $B_n(x, \delta)$ ($n \geq 0$), its image $g^n(G)$ is a $(D_x g^n E_x, D_x g^n F_x)$ -graph with dispersion $\leq c$.*

Proof Let $\beta = \min_{x \in M} m(D_x g)$. Since dominated splitting can be extended on the closure of Λ and dominated splitting is always continuous (see [1]), we can take a finite constant

$$\alpha = \sup_{x \in \Lambda} \gamma(E_x, F_x).$$

For given $c > 0$ and for the above α, β , take $\tau > 0$ satisfying Lemma 3.3. Since $D_x g$ is uniformly continuous on M , there is $\delta > 0$ such that if $d(x, y) < \delta$, one has

$$\|D_x g - D_y g\| \leq \tau.$$

By applying Lemma 3.3, we get the following:

Fact For any $y \in \Lambda$ and every (E_y, F_y) -graph H with dispersion $\leq c$ contained in the ball $B_\delta(y)$, its image $g(H)$ is a $((D_y g)E_y, (D_y g)F_y)$ -graph with dispersion $\leq c$.

We prove Lemma 3.4 by induction. The conclusion is trivial for $n = 0$. Assume it holds for some $n \geq 0$, that is, we assume that if G is a (E_x, F_x) -graph with dispersion $\leq c$ contained in Bowen ball $B_n(g, \delta, x)$ then $g^n(G)$ is a $(D_x g^n E_x, D_x g^n F_x)$ -graph with dispersion $\leq c$. Now let G is a (E_x, F_x) -graph with dispersion $\leq c$ contained in Bowen ball $B_{n+1}(g, \delta, x)$. Using $B_{n+1}(g, \delta, x) \subseteq B_n(g, \delta, x)$, G is also contained in $B_n(g, \delta, x)$. So, by assumption $g^n(G)$ is a $(D_x g^n E_x, D_x g^n F_x)$ -graph with dispersion $\leq c$. Take $y = g^n(x) \in \Lambda$ and let $H = g^n(G)$. Notice that

$$(D_x g^n E_x, D_x g^n F_x) = (E_{g^n x}, F_{g^n x}) = (E_y, F_y)$$

and

$$H = g^n(G) \subseteq g^n(B_n(g, \delta, x)) \subseteq B_\delta(g^n(x)) = B_\delta(y).$$

Thus H is a (E_y, F_y) -graph with dispersion $\leq c$ contained in $B_\delta(y)$. Using the above **Fact**, we have $g(H)$ is a $((D_y g)E_y, (D_y g)F_y)$ -graph with dispersion $\leq c$. Observe that

$$g^{n+1}(G) = g(H)$$

and

$$((D_x g^{n+1})E_x, (D_x g^{n+1})F_x) = ((D_y g)E_y, (D_y g)F_y),$$

we get that $g^{n+1}(G)$ is a $((D_x g^{n+1})E_x, (D_x g^{n+1})F_x)$ -graph with dispersion $\leq c$. \square

Now we are ready to prove Pesin's formula.

Proof of Theorem 2.2 Put

$$\Sigma_j = \{x \mid \dim F(x) = j\}$$

and let

$$S = \{j \geq 0 \mid \mu(\Sigma_j) > 0\}.$$

If $j \in S$, let μ_j be the measure on M given by

$$\mu_j(A) = \frac{\mu(A \cap \Sigma_j)}{\mu(\Sigma_j)}$$

for all Borel subset A of M . Then

$$\mu = \sum_{j \in S} \mu(\Sigma_j) \cdot \mu_j$$

and thus by the affine property of metric entropy we have

$$h_\mu(f) = \sum_{j \in S} \mu(\Sigma_j) h_{\mu_j}(f).$$

Thus, all we have to show is that

$$h_{\mu_j}(f) \geq \int \chi(x) d\mu_j.$$

This inequality obviously holds for $j = 0$. Suppose $j > 0$. Note that $\mu \ll \text{Leb}$ implies $\mu_j \ll \text{Leb}$ for all $j \in S$. Hence, to simplify the notation we put

$$\mu = \mu_j, \quad \Sigma = \Sigma_j.$$

Fix any $\varepsilon > 0$. Take N_0 so large that the set $\Sigma_\varepsilon = \{x \in \Sigma \mid m(x) \leq N_0\}$ has μ -measure larger than $1 - \varepsilon$. Let $N = N_0!$ and $g = f^N$, then the splitting $T_{\Sigma_\varepsilon} M = E \oplus F$ satisfies 1-dominated with respect to g :

$$\frac{\|Dg|_{E(x)}\|}{m(Dg|_{F(x)})} \leq \prod_{j=0}^{\frac{N}{m(x)}-1} \frac{\|Df^{m(x)}|_{E(f^{jm(x)}x)}\|}{m(Df^{m(x)}|_{F(f^{jm(x)}x)})} \leq \left(\frac{1}{2}\right)^{\frac{N}{m(x)}} \leq \frac{1}{2}, \quad \forall x \in \Sigma_\varepsilon.$$

Note that Σ_ε is f -invariant and thus g -invariant. In what follows, in order to avoid a cumbersome and conceptually unnecessary use of coordinate charts, we shall treat M as if it were a Euclidean space. The reader will observe that all our arguments can be easily formalized by a completely straightforward use of local coordinates.

Since dominated splitting can be extended on the closure of Σ_ε and dominated splitting is always continuous (see [1]), we can take and fix two constants $c > 0$ and $a > 0$ so small that if $x \in \Sigma_\varepsilon$, $y \in M$ and $d(x, y) < a$, then for every linear subspace $E \subseteq T_y M$ which is a $(E(x), F(x))$ -graph with dispersion $< c$ we have

$$\left| \log |\det D_y g|_E| - \log |\det(D_x g)|_{F(x)}| \right| < \varepsilon.$$

Thus

$$|\det D_y g|_E| \geq |\det(D_x g)|_{F(x)}| \cdot e^{-\varepsilon}. \quad (3.1)$$

By Lemma 3.4, there exists $\delta \in (0, a)$ such that for every $x \in \Sigma_\varepsilon$ and any (E_x, F_x) -graph G with dispersion $\leq c$ contained in the ball $B_n(g, \delta, x)$ ($n \geq 0$), its image $g^n(G)$ is a $((D_x g^n)E_x, (D_x g^n)F_x)$ -graph with dispersion $\leq c$.

Let ν be the Lebesgue measure on M . We give a claim as follows:

Claim. For every $x \in \Sigma_\varepsilon$,

$$h_\nu(g, \delta, x) \geq N\chi(x) - \varepsilon.$$

By Lemma 3.1, this property will imply that

$$h_\mu(g) \geq \int_M h_\nu(g, \delta, x) d\mu$$

$$\begin{aligned}
&\geq \int_{\Sigma_\varepsilon} h_\nu(g, \delta, x) d\mu \\
&\geq \int_{\Sigma_\varepsilon} (N\chi(x) - \varepsilon) d\mu \\
&= \int_M N\chi(x) d\mu - \int_{M \setminus \Sigma_\varepsilon} N\chi(x) d\mu - \varepsilon \cdot \mu(\Sigma_\varepsilon) \\
&\geq \int_M N\chi(x) d\mu - N \cdot C \cdot \dim(M) \cdot \mu(M \setminus \Sigma_\varepsilon) - \varepsilon \\
&\geq \int_M N\chi(x) d\mu - N \cdot C \cdot \dim(M) \cdot \varepsilon - \varepsilon
\end{aligned}$$

where $C = \max_{x \in M} \log \|D_x f\|$.

Hence,

$$h_\mu(f) = \frac{1}{N} h_\mu(g) \geq \int_M \chi(x) d\mu - C \cdot \dim(M) \cdot \varepsilon - \varepsilon.$$

Since ε is arbitrary this completes the proof of our theorem.

It remains to prove the claim. Fix any $x \in \Sigma_\varepsilon$. There exists $B > 0$ satisfying

$$\nu(B_n(g, \delta, x)) = B \int_{E(x)} \nu[(y + F(x)) \cap B_n(g, \delta, x)] d\nu(y)$$

for all $n \geq 0$, where ν also denotes the Lebesgue measure in the subspaces $E(x)$ and $y + F(x)$, $y \in E(x)$. Thus the claim is reduced to showing that

$$\limsup_{n \rightarrow +\infty} \inf_{y \in E(x)} \frac{1}{n} [-\log \nu(\Lambda_n(y))] \geq N\chi(x) - \varepsilon, \quad (3.2)$$

where

$$\Lambda_n(y) = (y + F(x)) \cap B_n(g, \delta, x).$$

If $\Lambda_n(y)$ is not empty, by Lemma 3.4 we have that

$$g^n(\Lambda_n(y)) \text{ is a } (E(g^n(x)), F(g^n(x)))\text{-graph with dispersion } \leq c.$$

Take $D > 0$ such that $D > \text{vol}(G)$ (where $\text{vol}(\cdot)$ denotes volume) for every $(E(w), F(w))$ -graph G with dispersion $\leq c$ contained in $B_\delta(w)$, $w \in \Sigma_\varepsilon$. Observe that

$$g^n(\Lambda_n(y)) \subseteq g^n B_n(g, \delta, x) \subseteq B_\delta(g^n(x)), \quad g^n(x) \in \Sigma_\varepsilon,$$

we have

$$D > \text{vol}(g^n(\Lambda_n(y))) = \int_{\Lambda_n(y)} |\det(D_z g^n)|_{T_z \Lambda_n(y)} d\nu(z).$$

Since

$$g^j(\Lambda_n(y)) \subseteq g^j B_n(g, \delta, x) \subseteq B_\delta(g^j(x)) \subseteq B_a(g^j(x)), \quad j = 0, 1, 2, \dots, n,$$

we have for any $z \in \Lambda_n(y)$,

$$d(g^j(z), g^j(x)) < a, \quad j = 0, 1, 2, \dots, n.$$

By inequality (3.1), we have

$$\begin{aligned} & |\det(D_z g^n)|_{T_z \Lambda_n(y)}| \\ &= \prod_{j=0}^{n-1} |\det(D_{g^j(z)} g)|_{T_{g^j(z)} g^j \Lambda_n(y)}| \\ &\geq \prod_{j=0}^{n-1} [|\det(D_{g^j(x)} g)|_{F(g^j(x))}| \cdot e^{-\varepsilon}] \\ &= |\det(D_x g^n)|_{F(x)}| \cdot e^{-n\varepsilon}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{n} \log D &\geq \frac{1}{n} \log \int_{\Lambda_n(y)} |\det(D_z g^n)|_{T_z \Lambda_n(y)} d\nu(z) \\ &\geq \frac{1}{n} \log \int_{\Lambda_n(y)} |\det(D_x g^n)|_{F(x)}| \cdot e^{-n\varepsilon} d\nu(z) \\ &= \frac{1}{n} \log [\nu(\Lambda_n(y)) \cdot |\det(D_x g^n)|_{F(x)}| \cdot e^{-n\varepsilon}] \\ &= \frac{1}{n} \log \nu(\Lambda(y)) + \frac{1}{n} \log |\det(D_x g^n)|_{F(x)}| - \varepsilon. \end{aligned}$$

It follows that

$$\lim_{n \rightarrow +\infty} -\frac{1}{n} \log \nu(\Lambda(y)) \geq \lim_{n \rightarrow +\infty} \frac{1}{n} \log |\det(D_x g^n)|_{F(x)}| - \varepsilon.$$

Combining this inequality and following equality from Oseledec theorem[9]

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log |\det(D_x g^n)|_{F(x)}| = N\chi(x),$$

we complete the proof of (3.2). This completes the proof of Theorem 2.2. \square

4 Proof of Theorem 2.5

In this section we prove Theorem 2.5. Before that we need a result of Bochi and Viana[2].

Theorem 4.1. ([2]) *There is a residual subset $\mathcal{R} \subseteq \text{Diff}_m^1(M)$ such that for every $f \in \mathcal{R}$ and for m a. e. $x \in M$, the Oseledec splitting of f is either trivial (i.e., all Lyapunov exponents are zero) or dominated at x .*

Proof of Theorem 2.5 Let $\mathcal{R} \subseteq \text{Diff}_m^1(M)$ be the same as in Theorem 4.1. Take and fix a diffeomorphism $f \in \mathcal{R}$. For m a. e. $x \in M$, we can define

$$\chi(x) = \sum_{\lambda_i(x) \geq 0} \lambda_i(x).$$

By Ruelle's inequality[11], we have

$$h_m(f) \leq \int \chi(x) dm.$$

Thus we only need to prove that

$$h_m(f) \geq \int \chi(x) dm.$$

Let

$$\Sigma_0 = \{x \in M \mid \text{the Oseledec splitting of } f \text{ is trivial at } x\}$$

and

$$\Sigma_1 = \{x \in M \mid \text{the Oseledec splitting of } f \text{ is dominated at } x\}.$$

Without loss of generality, we assume that $m(\Sigma_0) > 0$ and $m(\Sigma_1) > 0$. Let m_j be the measure on M given by

$$m_j(A) = \frac{m(A \cap \Sigma_j)}{m(\Sigma_j)} \quad (j = 0, 1)$$

for all Borel subset A of M . Then $m_0(\Sigma_0) = 1$, $m_1(\Sigma_1) = 1$. More precisely, for m_0 a. e. x , the Oseledec splitting is trivial at x and for m_1 a. e. x , the Oseledec splitting is dominated at x . Note that

$$m = m(\Sigma_0) \cdot m_0 + m(\Sigma_1) \cdot m_1.$$

Thus by the affine property of metric entropy we have

$$h_m(f) = m(\Sigma_0) \cdot h_{m_0}(f) + m(\Sigma_1) \cdot h_{m_1}(f).$$

Based on these analysis we only need to prove that

$$h_{m_i}(f) \geq \int \chi(x) dm_i, \quad i = 0, 1.$$

Since the metric entropy are always non-negative, obviously we have

$$h_{m_0}(f) \geq 0 = \int \chi(x) dm_0.$$

Note that m_1 are absolutely continuous relative to m . By Theorem 2.2, we get

$$h_{m_1}(f) \geq \int \chi(x) dm_1.$$

This completes the proof of Theorem 2.5. □

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